Vortex Flow over a Flat Surface with Suction

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Introduction

BY the use of momentum integral methods, Taylor[1] and Cooke[2] calculated the laminar boundary-layer development on the interior surface of a frustum of a cone, with a potential vortex as the outer flow. Taylor's swirl atomizer problem has also been examined by means of similarity transformations of the boundary-layer equations by Moore,[3] Mager,[4] and Rott and Lewellen,[5] and in Ref. 3 it is concluded that no valid solution can be found. However, in Ref. 4 it is noted that the valid solution may be found if the velocity profile of secondary flow changes sign somewhere in the boundary layer. Contrary to this, in Ref. 5 the existence of similarity solution is denied by a simple and clear consideration. Here, for simplicity we restrict our attention to the vortex flow over an infinite flat surface, i.e., to the case when the vertex angle of a cone is equal to 180°. For this problem, even the complete Navier-Stokes equations have a similarity solution only for a limited range of the Reynolds number.[6–8] In this Note, we attempt to solve the boundary-layer equations under the boundary condition of distributed suction, and show without any speculation to the singularity on the vortex axis that a formal similarity solution can be obtained for suction parameter greater than a certain value.

Analysis

The laminar similar boundary-layer equations for steady incompressible flow with a potential vortex as the outer flow are[5]

$$f''' + ff'' + f'^2 + g^2 - 1 = 0 \tag{1}$$

$$g'' + fg' = 0 \tag{2}$$

subject to the inner and outer boundary conditions

$$\begin{aligned} f(0) &= \sigma, \quad f'(0) = f'(\infty) = 0 \\ g(0) &= 0, \quad g(\infty) = 1 \end{aligned} \tag{3}$$

where primes attached to f and g denote differentiation with respect to $\zeta = (z/r)(C/\nu)^{1/2}$, ν is kinematic viscosity, and C/r is the swirl velocity of a potential vortex. The velocity components (u,v,w) in the directions of cylindrical coordinates (r,ϕ,z) are expressed as $u = (C/r)f'(\zeta)$, $v = (C/r)g(\zeta)$, and $w = (\zeta f' - f)(C\nu)^{1/2}/r$. The suction parameter σ is equal to $-w_0 r/(C\nu)^{1/2}$, where w_0 is the suction velocity at the surface. Since σ is set constant, w_0 is inversely proportional to r.

Equations (1) and (2) can be solved after the manner of Stuart.[9] We define $\eta = \sigma\zeta$ and introduce the following functions of η:

$$F(\eta) = f(\zeta)/\sigma, \quad G(\eta) = g(\zeta) \tag{4}$$

Substituting Eq. (4) into Eqs. (1) and (2), we have

$$F''' + FF'' + F'^2 + \epsilon(G^2 - 1) = 0 \tag{5}$$

$$G'' + FG' = 0 \tag{6}$$

where $\epsilon = \sigma^{-4}$, and primes attached to F and G denote differentiation with respect to η. Equation (3) becomes

$$\begin{aligned} F(0) &= 1, \quad F'(0) = F'(\infty) = 0 \\ G(0) &= 0, \quad G(\infty) = 1 \end{aligned} \tag{7}$$

For $\epsilon \ll 1$ (i.e., $\sigma^4 \gg 1$), a solution of Eqs. (5) and (6) is assumed of the form

$$\begin{aligned} F(\eta) &= F_0(\eta) + \epsilon F_1(\eta) + \epsilon^2 F_2(\eta) + \dots \\ G(\eta) &= G_0(\eta) + \epsilon G_1(\eta) + \epsilon^2 G_2(\eta) + \dots \end{aligned} \tag{8}$$

The substitution of Eq. (8) into Eqs. (5–7) yields a set of differential equations and boundary conditions for F_i and G_i, the solutions of the first of which are $F_0 = 1$, $G_0 = 1 - e^{-\eta}$. Higher-order terms can be obtained in succession by solving the linear differential equations. The solutions up to the second order are

$$F_1 = -\tfrac{7}{4} + (\tfrac{3}{2} + 2\eta)e^{-\eta} + \tfrac{1}{4}e^{-2\eta}$$

$$G_1 = (\tfrac{55}{24} - \tfrac{7}{4}\eta)e^{-\eta} - (\tfrac{9}{4} + \eta)e^{-2\eta} - \tfrac{1}{24}e^{-3\eta}$$

$$\begin{aligned} F_2 = {} & -\tfrac{2995}{576} + (-\tfrac{5}{3} + \tfrac{121}{24}\eta + \tfrac{7}{2}\eta^2)e^{-\eta} + (\tfrac{37}{6} + \tfrac{59}{8}\eta + \\ & 2\eta^2)e^{-2\eta} + (\tfrac{11}{16} + \tfrac{13}{36}\eta)e^{-3\eta} + \tfrac{7}{576}e^{-4\eta} \end{aligned}$$

$$\begin{aligned} G_2 = {} & (\tfrac{162049}{20736} - \tfrac{685}{576}\eta - \tfrac{49}{32}\eta^2)e^{-\eta} - (\tfrac{229}{96} + \tfrac{205}{24}\eta + \\ & \tfrac{7}{2}\eta^2)e^{-2\eta} - (\tfrac{3011}{576} + \tfrac{445}{96}\eta + \eta^2)e^{-3\eta} - \\ & (\tfrac{259}{1296} + \tfrac{5}{54}\eta)e^{-4\eta} - \tfrac{5}{2304}e^{-5\eta} \end{aligned}$$

From these results, two components of wall shear stress $\tau_r = \mu(\partial u/\partial z)_{z=0}$, $\tau_\phi = \mu(\partial v/\partial z)_{z=0}$, and the total volume of secondary flow

$$Q = 2\pi r \int_0^\infty u dz$$

can be calculated to yield

$$\begin{aligned} -\tau_r/[\rho(C/r)^2(\nu/C)^{1/2}] &= \tfrac{3}{2}\sigma^{-1} + \tfrac{197}{144}\sigma^{-5} + O(\sigma^{-9}) \\ \tau_\phi/[\rho(C/r)^2(\nu/C)^{1/2}] &= \sigma - \tfrac{5}{12}\sigma^{-3} - \tfrac{97}{96}\sigma^{-7} + O(\sigma^{-11}) \\ -Q/2\pi r(C\nu)^{1/2} &= \tfrac{7}{4}\sigma^{-3} + \tfrac{2995}{576}\sigma^{-7} + O(\sigma^{-11}) \end{aligned} \tag{9}$$

where $\mu = \rho\nu$ and ρ is density of fluid.

The solutions of Eqs. (1–3) were obtained for $\sigma^4 \gg 1$. However, since there is no solution at $\sigma = 0$, we have to solve these equations for moderate values of σ and clarify the limit of σ, below which no solution exists. This is executed numerically by the method by Nachtsheim and Swigert[10] on a NEAC-2200-500 computer at the Computing Center of the Tohoku University. It is found that the solutions are obtainable only for $\sigma \geqq 1.74$. For this range of σ, the radial velocity profiles are shown in Fig. 1 as compared with those from Eq. (8). For $\sigma \geqq 3$, the difference between two is indistinguishable visually. For circumferential velocity profiles, the difference between two is sufficiently small even at

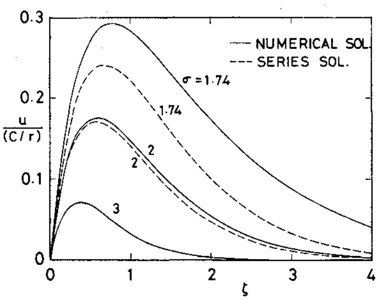

Fig. 1 Radial velocity profiles.

Table 1 Boundary-layer quantities, compared with those from Eq. (9) which are in parentheses

σ	$-\tau_r/[\rho(C/r)^2(\nu/C)^{1/2}]$		$\tau_\phi/[\rho(C/r)^2(\nu/C)^{1/2}]$		$-Q/2\pi r(C\nu)^{1/2}$	
1.74	1.068	(0.9478)	1.602	(1.640)	0.6991	(0.4399)
2	0.8076	(0.7928)	1.937	(1.940)	0.2795	(0.2594)
3	0.5059	(0.5056)	2.984	(2.984)	0.06735	(0.06719)

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$\sigma = 1.74$. The boundary-layer quantities obtained numerically are given in Table 1 as compared with those from Eq. (9).

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Shock and Vibration Isolation Using a Nonlinear Elastic Suspension

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SHOCK and vibration problems in the aerospace and transportation industries arise from many causes, such as the isolation of instruments and controls or the protection of human occupants of vehicles. The usual solution to these problems involves the use of lightly damped flexible supports. These soft supports cause the natural frequency of the suspension system to be far below the disturbing frequency. This solution is effective for the isolation of steady-state vibration; however, when these suspensions encounter shock excitation their softness often leads to damagingly large deflections. It has been pointed out that this undesirable feature is not present in suspension systems utilizing symmetrically nonlinear springs that harden.[1–3] These springs become progressively stiffer when subjected to large deflections from their "operating point." A thorough discussion of analysis methods for systems having nonlinear springs has been made by Stoker.[4]

A number of ingenious ways have been developed to produce nonlinear spring devices,[5,6] but unfortunately many of these are not symmetrical in behavior or are rather complex to construct. As weight, cost, and reliability requirements become more important, designers are forced to search for new ways to improve existing designs by reducing the number of moving parts in suspension systems. Because of its simple construction and symmetrically hardening behavior, the device shown in Fig. 1a holds much promise as a shock and vibration isolation mount. The purpose of this Note is to present information for use in the design of suspension systems utilizing this device.

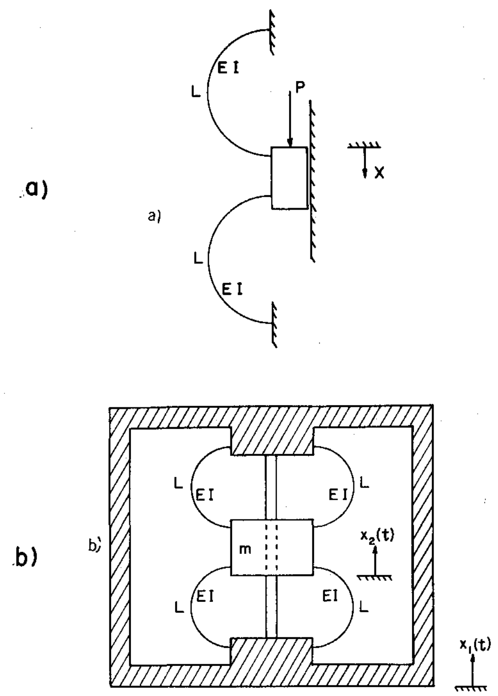

Fig. 1 The "elastica" suspension spring: a) single spring pair; b) two spring pairs in parallel.

Elastica Suspension Spring

The device shown in Fig. 1a consists of a pair of flexible strips each having length L, elastic modulus E, and cross section moment of inertia I. These two initially straight strips are each clamped in a semicircular shape. As the center platform is forced downward the upper flexible strip deflects into a shape called the "nodal elastica" and the lower strip deflects into a shape called the "undulating elastica." The describing equations for these "elastica" curves are well known[7]

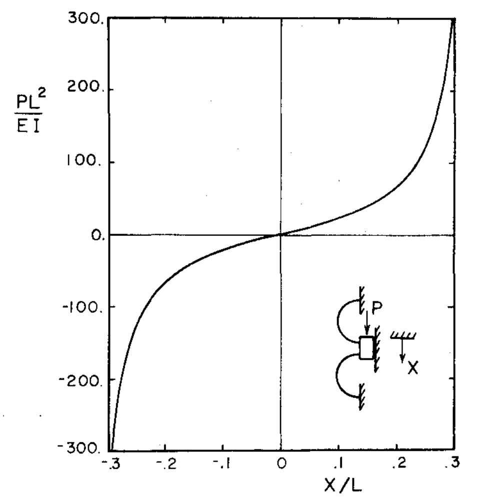

Fig. 2 Nondimensional load-vs-deflection curve for the "elastica" suspension spring pair.

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